SoS Certifiability of Subgaussian Distributions and its Algorithmic Applications

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Simons Institute (UC Berkeley) $\longrightarrow$ CMU Statistics

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Information-Computation Tradeoffs for Statistical Problems

Joint work with

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Outline

Motivation: Distributional assumptions and accuracy

Generic Estimation Problem. Let $\mathcal{P}$ be a family of distributions over $\mathbb{R}^d$ and $\theta^* : \mathcal{P} \to \Theta$ be the target parameter.

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Output: $\widehat{\theta}$ such that distance$\left(\widehat{\theta}, \theta^*(Q)\right)$ is small w.h.p.

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Understanding "niceness" versus "tractable niceness"

Motivation: Tail decay and moment bounds

- Often, the optimal error is governed by **tail bounds** of **linear forms**:

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- This motivates the definition of **subgaussian distributions**

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For all $v \in \mathbb{R}^d$ and even t: $\mathbb{E}\left[\langle v, X - \mu \rangle^t\right] \leqslant \mathbb{E}_{G \sim \mathcal{N}(0, I_d)}[\langle v, G \rangle^t]$

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Algorithmically-tractable version of subgaussianity?

Example: Robust covariance estimation of sub-Gaussians

Let $\mathcal{Q}$ be a **sub-Gaussian** distribution over $\mathbb{R}^d$ with covariance Σ

Input: outlier-corrupted samples from $\mathcal{Q}$

a small fraction of samples are arbitrary

Output: $\widehat{\Sigma}$ such that $\|\widehat{\Sigma} - \Sigma\|_{\mathrm{op}}$ is small w.h.p.

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What are the underlying algorithmic challenges?

Challenges in robust covariance estimation

Algorithmic template: robust **mean** estimation

1. While there exists a direction v with large **variance**:
   1.1 Filter a point x if $\langle v, x \rangle$ is too large
2. **return** sample mean

Challenges in robust covariance estimation

Algorithmic template: robust **covariance** estimation

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Do sub-Gaussian data (X) have algorithm-friendly structure?

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  - Under Exponential Time Hypothesis [BBHKSZ12]
  - Stronger hardness under Small Set Expansion Hypothesis [BBHKSZ12; HL19]

[BBHKSZ12] B. Barak, F. G.S.L. Brandao, A. W. Harrow, J. Kelner, D. Steurer, Y. Zhou. Hypercontractivity, sum-of-squares proofs, and their applications.

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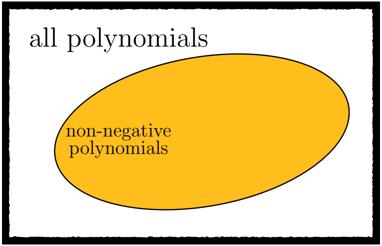
- It is **hard** to check if it is always $\geqslant 0$



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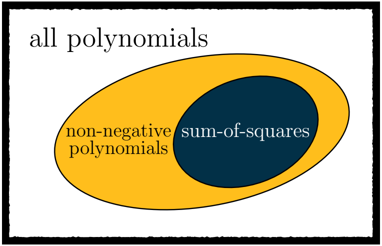

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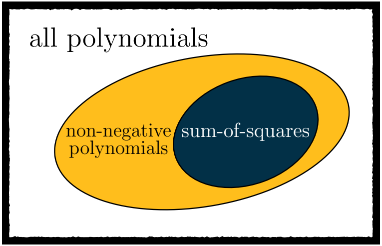
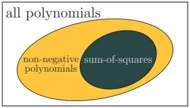
a semidefinite program





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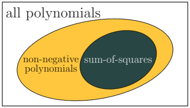
Definition. (Algorithm-friendly sub-Gaussians) X is **certifiably-sub-Gaussian** if

not only are the moments bounded

$$\mathbb{E}_{G\sim\mathcal{N}(0,\mathbf{I})}\left[\langle v, G\rangle^t\right] - \mathbb{E}_X\left[\langle v, X\rangle^t\right] \geqslant 0$$



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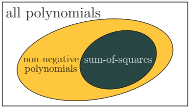
not only $\geqslant 0$,
but also a SoS polynomials

for some polynomials $q_1(\cdot), \ldots,$

sub-Gaussian: moments less than Gaussians



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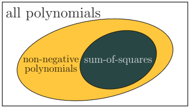

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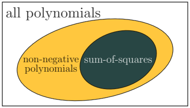

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Which distributions are certifiably sub-Gaussian anyway?

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  - rotational invariance, independent coordinates
  - log-Sobolev distributions (more generally, Poincare) [KSS18]
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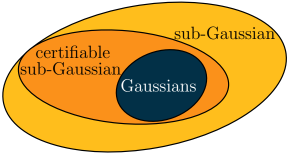

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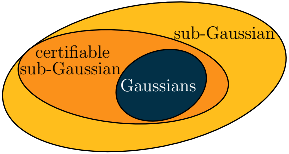

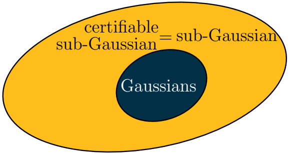

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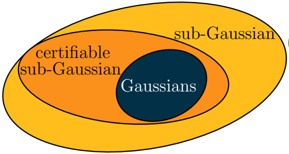
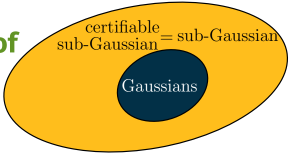


Can we characterize the class of certifiably sub-Gaussians?



Outline

Our result: Certifiability of sub-Gaussian distributions

Theorem: [Diakonikolas-Hopkins-**P**-Tiegel]

All sub-Gaussian distributions are **certifiably** so! (upto an absolute constant C)

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[DHPT25] I. Diakonikolas, S. Hopkins, A. Pensia, S. Tigel. SoS Certifiability of Subgaussian Distributions & Its Algorithmic Applications. *STOC*. 2025

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- Corollary: new polynomial-time algorithms for sub-Gaussian data
- Corollary: **new computational lower bounds** for Gaussian data (certification)

[DHPT25] I. Diakonikolas, S. Hopkins, A. Pensia, S. Tigel. SoS Certifiability of Subgaussian Distributions & Its Algorithmic Applications. *STOC*. 2025

New algorithmic guarantees for sub-Gaussian data

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- We give the **first polynomial-time algorithms** for
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- Our algorithmic guarantees are qualitatively-**optimal*** (within low-degree polynomial tests, statistical query algorithms, sum-of-squares hierarchy)

New algorithmic guarantees for sub-Gaussian data

Robust Estimation Task	Inlier Distribution	Information-theoretic Error	Previous Best Guarantee in Polynomial Time	New Guarantees
Covariance estimation: Relative spectral norm	hypercontractive sub-Gaussian	$\widetilde{\Theta}(\epsilon)$	No general algorithm	$\epsilon^{1-2/m}$
Mean estimation: Mahalanobis norm	hypercontractive sub-Gaussian	$\widetilde{\Theta}(\epsilon)$	No general algorithm	$\epsilon^{1-1/m}$
Linear regression: Arbitrary noise	hypercontractive sub-Gaussian	$\widetilde{\Theta}(\epsilon)$	No general algorithm	$\epsilon^{1-\frac{2}{m}}$
Mean estimation: Euclidean norm	sub-Gaussian	$\widetilde{\Theta}(\epsilon)$	$\sqrt{\epsilon}$	$\epsilon^{1-1/m}$
List-decodable mean estimation	sub-Gaussian	$\widetilde{\Theta}(\epsilon)$	$\sqrt{\frac{1}{1-\epsilon}}$	$\left(\frac{1}{1-\epsilon}\right)^{-\Omega\left(\frac{1}{m}\right)}$
Mixture models: Mixture of k Δ-separated components	Each component is sub-Gaussian	$\Delta \gtrsim \sqrt{\log k}$	$\Delta \gtrsim k^{\Omega(1)}$	$\Delta \gtrsim k^{O\left(\frac{1}{m}\right)}$

Outline

- Motivation
- Prior Work
- Our Result
- **Proof Sketch**
- Conclusion

Proof sketch

A series of equivalences using

1. duality
2. empirical processes
3. linearization

$q(v) := \mathbb{E}[\langle v, X\rangle^t]$
$B := (\Theta(\sqrt{t}))^t$

Proof sketch: (1/3) Duality

Original formulation

Certifiable formulation

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Depends only on the degree-t moments of $\mathbb{E}$

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A "pseudo"-expectation

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Set of degree-t moments over $\mathbb{R}^p$

Certifiable formulation

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Set of degree-t $\widetilde{\mathbb{E}}$

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A "pseudo"-expectation

Equivalent to showing that the dual is bounded

certifiabile sub-Gaussianity $\iff$ dual is bounded

Proof sketch: (2/3) Empirical process

$\widetilde{\mathbb{E}}$: a "pseudo"-expectation

$$\sup_{\widetilde{\mathbb{E}}\text{:degree-}t} \frac{\widetilde{\mathbb{E}}\left[\mathbb{E}[\langle v, X\rangle^t]\right]}{\widetilde{\mathbb{E}}\left[\|v\|^t\right]}$$

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Is this empirical process bounded for **all** sub-Gaussians?

Proof sketch: (3/3) Chaining

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  - all sub-Gaussian **linear** process $\leqslant$ Gaussian **linear** process

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For all sub-Gaussians, our process is bounded by Gaussians!

Proof sketch: Putting the pieces together

X is certifiably-sub-Gaussian

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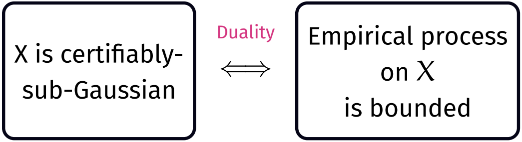

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X is certifiably-sub-Gaussian $\overset{\text{Duality}}{\iff}$ Empirical process on X is bounded $\overset{\text{Linearization}}{\iff}$ Linearized process on X is bounded

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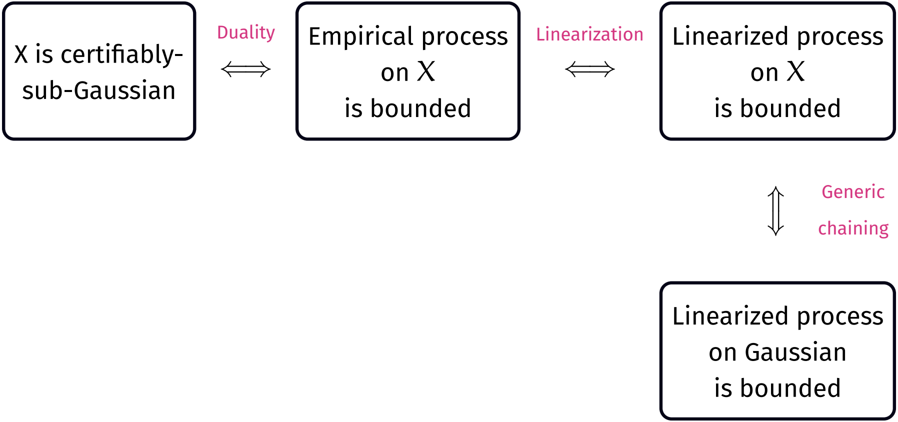

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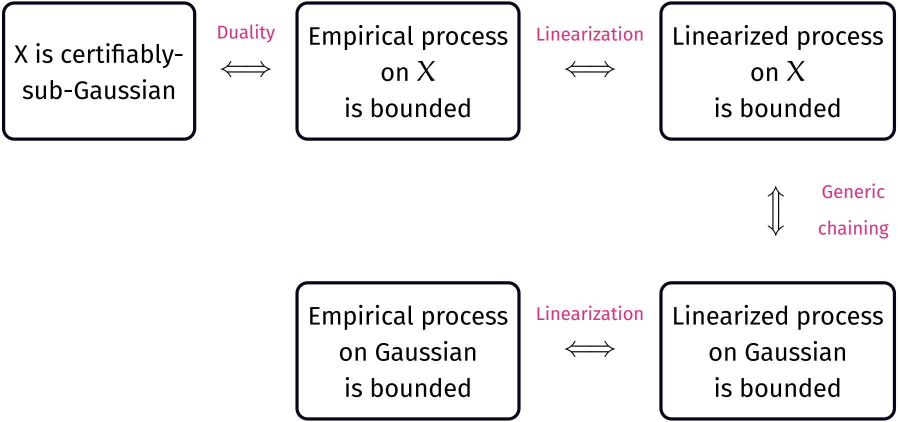

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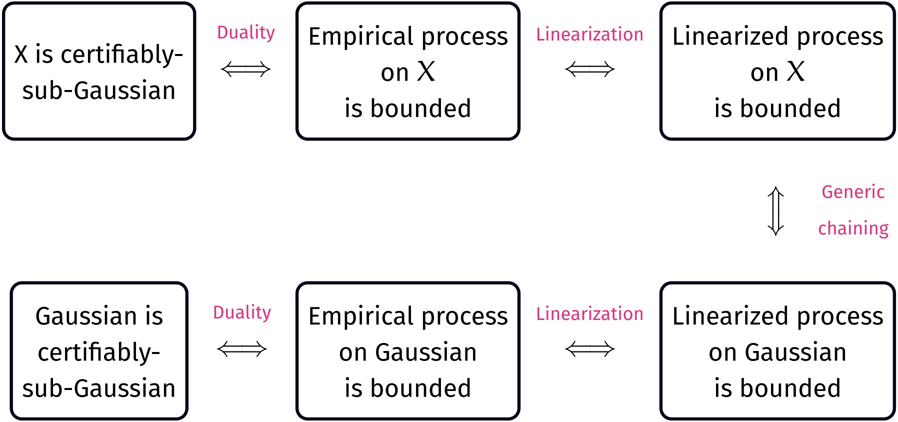

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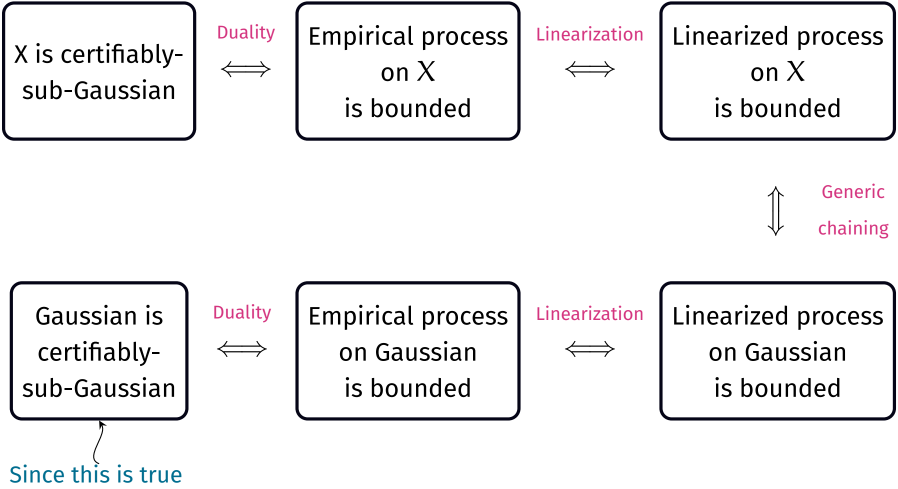

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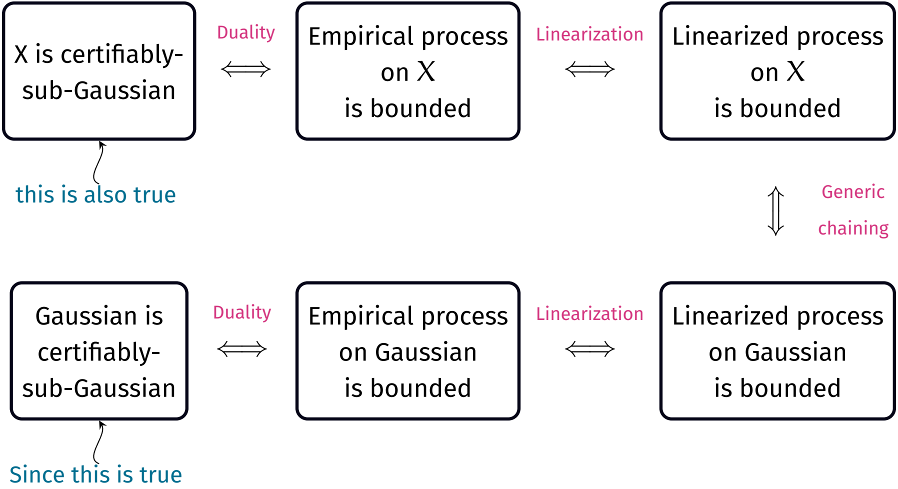

Our result: Certifiability of sub-Gaussian distributions

Theorem: [Diakonikolas-Hopkins-**P**-Tiegel]

All sub-Gaussian distributions are **certifiably** so! (upto an absolute constant C)

There exists C s.t. for any $d \in \mathbb{N}$, sub-Gaussian distribution X on $\mathbb{R}^d$, and even $t \in \mathbb{N}$,

$$\mathbb{E}_{G \sim \mathcal{N}(0, CI)} \left[\langle v, G\rangle^t\right] - \mathbb{E}_X \left[\langle v, X - \mu\rangle^t\right] = \sum_i q_i^2(v) \qquad \text{for some polynomials } q_1, \ldots,.$$

- Answers open questions of Kothari-Steinhardt (2018), Hopkins (2019) & extends Barak-Brandao-Harrow-Kelner-Steurer-Zhou (2012)
  - Even for the fourth moment ($t = 4$), the prior best bound was $\sqrt{d}$
- We show sub-Gaussians also have other algorithm-friendly structures
- Conceptual contribution: connecting SDP relaxations & empirical processes
- Corollary: new polynomial-time algorithms for sub-Gaussian data
- Corollary: **new computational lower bounds** for Gaussian data (certification)

[DHPT25] I. Diakonikolas, S. Hopkins, A. Pensia, S. Tigel. SoS Certifiability of Subgaussian Distributions & Its Algorithmic Applications. *STOC*. 2025

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Null: $X_1, \ldots, X_n$ are iid from $P \in \mathcal{P}$

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Thank you for your attention!

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Chaining and linearization

Talagrand's generic chaining. For any linear function class $\mathcal{F}_{\text{lin}}$

$$\mathbb{E}\Big[\sup_{(f_1,\dots,f_n)\in\mathcal{F}_{\text{lin}}}\tfrac{1}{n}\sum_{i=1}^{n} f_i(X_i)\Big] \lesssim \mathbb{E}\Big[\sup_{(f_1,\dots,f_n)\in\mathcal{F}_{\text{lin}}}\tfrac{1}{n}\sum_{i=1}^{n} f_i(G_i)\Big]$$

Our linearization lemma

Lemma. (f is linear-ish) For every $f \in \mathcal{F}$, there exists $\mathcal{G}$ such that

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- Extends to $\widetilde{\mathbb{E}}$: (i) $y_i = \langle v, X_i\rangle$ and (ii) Holder's inequality holds for $\widetilde{\mathbb{E}}$